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On the nonlinear stability of slowly varying time-dependent viscous flows

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In this paper we investigate the manner in which finite-amplitude disturbances are set up in viscous fluid flows that are changing slowly in time. It is shown that, when the appropriate Reynolds or Rayleigh number is slowly increased, then, no matter how slowly this change takes place, there is always a short time interval where a quasi-steady approach breaks down. In this time interval a finite-amplitude solution is set up which ultimately approaches that predicted by a quasi-steady theory. In order to demonstrate our ideas we discuss the Taylor-vortex problem in a situation in which the speed of the inner cylinder changes slowly in time. In particular we discuss the case when the speed of the inner cylinder is modulated slowly in time and it is found that at low frequencies the disturbances of most physical relevance are not periodic solutions of the equations of motion.

1. Introduction

Our concern is with the stability of viscous fluid flows that change slowly in time. In order to obtain precise results for a particular problem we concentrate on the Taylor-vortex problem with a time-dependent basic flow, but the method we use applies to convective- and parallel-flow instabilities. In particular we focus our attention on the problem investigated experimentally by Donnelly (1964) and theoretically by Hall (1975) and Riley & Lawrence (1976). Donnelly performed experiments in which the inner cylinder of the Taylor-vortex apparatus rotates with angular velocity $\Omega\{1+\epsilon \cos \omega t\}$, where t denotes time. The enhancement of stability by modulation found by Donnelly was not predicted by the subsequent theoretical investigations of Hall (1975) and Riley & Lawrence (1976). In this paper we shall show that at low frequencies some qualitative agreement between theory and experiment can be obtained by seeking finite-amplitude disturbances that are not periodic in time. We shall argue that for a basic time-periodic flow the disturbances of most physical relevance when the flow changes slowly in time are not periodic solutions of the equations of motion.

The latter assertion was also discussed by Rosenblat & Herbert (1970), who studied the linear stability of a fluid layer confined between boundaries having time-dependent temperatures. However, in the absence of nonlinear effects the role of the non-periodic disturbances could not be ascertained. The possible role of nonlinear effects in such problems was discussed by Davis & Rosenblat (1977) in the context of a model problem. Some of the ideas used in the present work are similar to those to be found in the latter paper.

The application of weakly nonlinear stability theory to viscous fluid flows is now a routine procedure (see e.g. Stuart 1971). This perturbation type of approach is necessarily valid only a small values of the disturbance amplitude close to the critical value λ_c of the appropriate stability parameter λ of the flow. Typically it is found that in an ϵ-neighbourhood of λ_c the time-dependent amplitude of a disturbance of magnitude $O(\epsilon^{\frac{1}{2}})$ satisfies an amplitude equation of the form

$$\frac{dA}{d\tau} = \mu\lambda_1 A + \gamma A|A|^2,$$

where τ is a slow time variable scaled on ϵ whilst $\lambda_1 = (\lambda - \lambda_c)/\epsilon$ and μ, γ are constant. The presence of imperfections and/or end effects introduces quadratic or constant terms on the right-hand side of the amplitude equation (see e.g. Hall & Walton 1979). It is easy to show that equilibrium solutions of the above amplitude equation bifurcate from $\lambda_1 = 0$. These solutions of course depend on λ_1, γ, and by studying their stability properties it can be inferred how the flow adjusts when λ_1 is varied. The implicit assumption made is that if λ_1 is varied slowly enough then it can be treated as a constant in the appropriate amplitude equation. We shall show that this is not the case and that, however slowly λ_1 varies, there is always a time interval near $\lambda_1 = 0$ where the appropriate amplitude equation is non-autonomous. If, as is invariably the case in centrifugal or convective instability problems, the real part of γ is negative, then in this time interval non-zero perturbations to the basic state ultimately approach the equilibrium state predicted by the quasi-steady theory. The quasi-steady approach therefore fails at the bifurcation point $\lambda_1 = 0$ so that it does not describe correctly the manner in which finite-amplitude motions develop when the flow becomes supercritical. A detailed investigation of the neighbourhood of $\lambda_1 = 0$ gives a long time structure consistent with the quasi-steady theory in the present problem. In more complicated problems where two or more modes are possible near $\lambda_1 = 0$ the latter result is not to be expected since there will in general be several possible states available. The particular state set up when the flow becomes supercritical will depend crucially on both how the flow is changing and the disturbance which triggers the instability. Some experimental results consistent with this prediction have recently been given by Donnelly & Park (1982).

In the present paper we consider in detail the nonlinear stability of low-frequency modulated circular Couette flow. We restrict our attention to the case when σ, the non-dimensional frequency of the modulation, is small compared with ϵ, the amplitude of the modulation. In a previous paper, using the method devised by DiPrima & Stuart (1973, 1975), Hall (1975) calculated finite-amplitude periodic Taylor vortices appropriate to the limit $\epsilon \to 0$ with σ/ϵ held fixed. In this paper we argue that if $\epsilon \to 0$ with $\sigma/\epsilon \ll 1$ then the disturbances of most physical relevance are not periodic solutions of the equations of motion. However, periodic disturbances still exist in this limit, but they lead to vortices that are exponentially small for a finite time interval during which the flow is locally supercritical. We believe that at sufficiently small values of σ such a result is not physically acceptable because random disturbances always present in any experiment will trigger the instability whenever the flow is locally supercritical.

We shall use our non-periodic solutions of the equations of motion to show how the amplitude of the Taylor vortex varies with Ω. Our results show that initially the amplitude grows linearly with Ω, but ultimately adjusts to grow like $(\Omega - \Omega_c)^{\frac{1}{2}}$, where Ω_c is the unmodulated critical angular velocity of the inner cylinder. This change in the manner in which the amplitude develops with increasing Ω is evident in some of Donnelly's experimental results.

The procedure adopted in this paper is as follows. In §2 we show how finite-amplitude perturbations to slowly varying circumferential flows can be calculated. In §3 we apply the results of §2 to modulated circular Couette flow. In §4 we show how our results are related to those previously obtained by using the approach of DiPrima & Stuart (1973, 1975).

2. The slowly varying Taylor-vortex problem

Consider the viscous flow between two concentric cylinders of radii R_1 and R_1+d driven by the motion of the inner cylinder, which has angular velocity $\Omega\{1+\epsilon f(\omega t)\}$. Here t denotes time, and ϵ and $\sigma = \omega d^2/\nu$ are taken to be small. The parameter σ represents the square of the ratio of d to the Stokes-layer thickness associated with an oscillatory viscous flow of frequency ω, so that if $\sigma \ll 1$ the flow responds in a quasi-steady manner. More precisely we assume that

$$1 \gg \epsilon \gg \sigma,$$

in contrast with the situation discussed by Hall (1975), where, amongst other limits, the case $\epsilon \sim \sigma$ was examined using the method of DiPrima & Stuart (1975).

Suppose next that, using d as a typical lengthscale, we introduce dimensionless radial and axial variables ζ and ϕ and then define $\tau = \omega t$. If u and v are suitable scaled radial and azimuthal disturbance-velocity components, then, using the notation of Hall (1975), the nonlinear partial differential equations to determine u and v in the small-gap limit are

$$\left(\mathscr{L}-\sigma\frac{\partial}{\partial\tau}\right)\mathscr{L}u = T\overline{V}\frac{\partial^2 v}{\partial\phi^2}-\frac{1}{2}\frac{\partial^2 Q_1}{\partial\phi^2}+\frac{1}{2}\frac{\partial^2 Q_2}{\partial\zeta\,\partial\phi}, \tag{1a}$$

$$\left(\mathscr{L}-\sigma\frac{\partial}{\partial\tau}\right)v = -u\frac{\partial\overline{V}}{\partial\zeta}-\tfrac{1}{2}Q_3, \tag{1b}$$

$$\frac{\partial u}{\partial\zeta}+\frac{\partial w}{\partial\phi} = 0, \tag{1c}$$

where

$$T = 2\Omega^2 R_1 d^3\nu^{-2} \tag{2}$$

is the Taylor number and

$$\mathscr{L} \equiv \frac{\partial^2}{\partial\zeta^2}+\frac{\partial^2}{\partial\phi^2}.$$

The nonlinear functions Q_1, Q_2, Q_3 are as defined by (2.5) of Hall (1975) and we note that for small σ the basic velocity field $\overline{V}$ can be expanded in the form

$$\overline{V} = 1-\zeta+\epsilon(1-\zeta)f(\tau)+O(\epsilon\sigma). \tag{3}$$

We now write T in the form

$$T = T_0+\epsilon T_1,$$

where T_0 is the critical Taylor number of linear stability theory whilst T_1 determines the elevation of T above T_0. In such a configuration linear theory would predict growth (or damping) rates of order ϵ/σ, so it is necessary to define the time variable $\bar{\tau}$ by

$$\bar{\tau} = \frac{\epsilon}{\sigma}\tau,$$

and then $\sigma\,\partial/\partial\tau$ in (1) must be replaced by $\sigma\,\partial/\partial\tau+\epsilon\,\partial/\partial\bar{\tau}$. Since we have taken $\sigma \ll \epsilon$ the latter expression is dominated by the 'fast' derivative $\epsilon\,\partial/\partial\bar{\tau}$. This means that the disturbance is changing rapidly on the $\bar{\tau}$ timescale compared to the basic flow. We then expand u, v in the form

$$\begin{aligned} u &= \epsilon^{\frac{1}{2}}u_0(\bar{\tau},\tau,\zeta)\cos a\phi+\epsilon u_1(\bar{\tau},\tau,\zeta)\cos 2a\phi+\epsilon^{\frac{3}{2}}u_2(\bar{\tau},\tau,\zeta)\cos a\phi+\dots,\\ v &= \epsilon^{\frac{1}{2}}v_0(\bar{\tau},\tau,\zeta)\cos a\phi+\epsilon v_1(\bar{\tau},\tau,\zeta)\cos 2a\phi+\epsilon v_{\mathrm{M}}(\bar{\tau},\tau,\zeta)+\epsilon^{\frac{3}{2}}v_2(\bar{\tau},\tau,\zeta)+\dots. \end{aligned} \tag{4}$$

Here a is the critical wavelength of linear theory, and the higher-order terms not shown explicitly will depend on σ. The determination of u_0 etc. is a routine calculation, and at first order we find that (u_0, v_0) can be written in the form

$$(u_0, v_0) = A(\bar{\tau}, \tau)\,(f_0(\zeta), g_0(\zeta)), \tag{5}$$

where A is an amplitude function to be determined at higher order whilst (f_0, g_0) is the eigenfunction of linear stability theory. In fact at order $\epsilon^{\frac{3}{2}}$ we find that A satisfies the equation

$$\frac{\partial A}{\partial \bar{\tau}} = \Gamma\left[f(\tau)+\frac{T_1}{2T_0}\right]A + a_1 A^3, \tag{6}$$

where the constants Γ and a_1 have the numerical values $+26$ and -10. If $f(\tau)+T_1/2T_0$ is positive then (6) has the 'locally equilibrated' solution

$$A^2 = \frac{-\Gamma}{a_1}\left[f(\tau)+\frac{T_1}{2T_0}\right], \tag{7}$$

whilst if $f(\tau)+T_1/2T_0$ is negative the corresponding solution is

$$A = 0. \tag{8}$$

Moreover any small disturbance to the equilibrium states (7) or (8) decays to zero on the $\bar{\tau}$-scale. Thus, to the order to which we have proceeded, the τ-dependence of the disturbance is passive. Thus (7) confirms the usual implicit assumption that a weakly nonlinear theory ignoring the slow time dependence of the basic flow will give an accurate prediction of the instantaneous (on the τ-scale) equilibrium configuration. Clearly at higher order this will not be the case, but we do not address that problem. Instead we investigate the relationship between the solutions (7) and (8) when the flow is locally neutral so that $f(\tau)+T_1/2T_0 = 0$. We shall see that the switch between these solutions does not take place in a quasi-steady manner.

Suppose then that $f(\tau)+T_1/2T_0$ vanishes at $\tau = \tau^*$, so that near $\tau = \tau^*$

$$u \sim \epsilon^{\frac{1}{2}}(\tau-\tau^*)^{\frac{1}{2}}\,[f'(\tau^*)]^{\frac{1}{2}}, \tag{9}$$

and at this stage we assume $f'(\tau^*) \neq 0$. We also note that the growth rate according to a quasi-steady linear stability analysis is then $O(\epsilon)\,(\tau-\tau^*)$. Thus if we choose to work in a $(\sigma/\epsilon)^{\frac{1}{2}}$ neighbourhood of τ this growth rate and the rate of change of the basic flow are comparable. We are therefore led to define the new time variable $\tilde{\tau}$ by

$$\tilde{\tau} = (\tau-\tau^*)\left(\frac{\epsilon}{\sigma}\right)^{\frac{1}{2}}, \tag{10}$$

and then $\sigma\,\partial/\partial\tau$ in (1) must be replaced by $(\epsilon\sigma)^{\frac{1}{2}}\,\partial/\partial\tilde{\tau}$. In view of (9), (10) we expect the disturbance to be $O(\epsilon^{\frac{1}{4}}\sigma^{\frac{1}{4}})$ in this regime. We therefore expand u in the form

$$u = (\epsilon\sigma)^{\frac{1}{4}}\tilde{u}_0(\tilde{\tau}, \zeta)\cos a\phi + (\epsilon\sigma)^{\frac{1}{2}}\tilde{u}_1(\tilde{\tau}, \zeta)\cos 2a\phi + (\epsilon\sigma)^{\frac{3}{4}}\tilde{u}_3(\tilde{\tau}, \zeta)\cos a\phi + O(\epsilon\sigma)^{\frac{3}{4}},$$

together with a similar expansion for v. The determination of $\tilde{u}_0$ etc. follows the usual procedure, and at $O(\epsilon\sigma)^{\frac{1}{4}}$ we find

$$\tilde{u}_c = B(\tilde{\tau})\,(f_0(\zeta), g_0(\zeta)),$$

where (f_0, g_0) is as defined earlier and $B(\tilde{\tau})$ is an amplitude function, which at $O(\epsilon\sigma)^{\frac{3}{4}}$ is found to satisfy

$$\frac{dB}{d\tilde{\tau}} = \Gamma f'(\tau^*)\,\tilde{\tau}B + a_1 B^3, \tag{11}$$

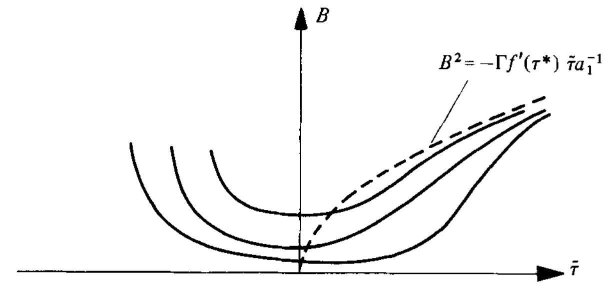

FIGURE 1. Some solutions of (11) for different values of B^+.

and depending on whether the flow becomes unstable or stable with τ increasing through $\tau = \tau^*$ the constant part of the coefficient of the linear term in (11) is positive or negative.

The equation (11) is easily integrated to give

$$B^2(\tilde{\tau}) = \frac{\exp\left[\Gamma f'(\tau^*)\,\tilde{\tau}^2\right]}{\dfrac{e^{\Gamma f'(\tau^*)\,\tau^{+2}}}{B^{+2}} - 2a_1 \displaystyle\int_{\tau^+}^{\tilde{\tau}} \exp\left[\Gamma f'(\tau^*)\,\theta^2\right] d\theta}, \tag{12}$$

where $B^+ = B(\tilde{\tau} = \tau^+)$ for some τ^+. If $f'(\tau^*) > 0$ we are interested in the solution of (12) for $\tilde{\tau} > \tau^+$, and (12) determines the development of an initial disturbance B^+ imposed at $\tau = \tau^+$. It follows directly from (12) that for any such disturbance with $B^+ \neq 0$

$$B \sim \pm(-\Gamma_{a_1}^{-1} f'(\tau^*)\,\tilde{\tau})^{\frac{1}{2}} \quad \text{as} \quad \tilde{\tau} \to \infty. \tag{13}$$

Some typical solutions having this property are illustrated in figure 1. The asymptotic form (13) is recognized to be the limit as $\tau \to \tau_+^*$ of the quasi-steady solution (7). Thus any small disturbance in a $\{\sigma/\epsilon\}^{\frac{1}{2}}$ neighbourhood of $\tau = \tau^*$ develops smoothly into the quasi-steady solution (7). These disturbances cannot be related to infinitesimal disturbances appropriate to linear theory because, for any B^+, τ^+, B will always develop a square root singularity when the denominator in (12) vanishes. Thus the disturbances which we discuss here are not connected to infinitesimal disturbances appropriate to the limit $\tilde{\tau} \to -\infty$.

Now let us consider the case when $f'(\tau^*) < 0$, so that the flow is unstable before $\tau = \tau^*$. In this case we are interested in a solution of (11) that matches with the quasi-steady solution (7) when $\tilde{\tau} \to -\infty$. The appropriate form of (12) is

$$B^2(\tilde{\tau}) = \frac{\exp\left[\Gamma f'(\tau^*)\,\tilde{\tau}^2\right]}{2a_1 \displaystyle\int_{\tilde{\tau}}^{-\infty} \exp\left[\Gamma f'(\tau^*)\,\theta^2\right] d\theta}, \tag{14}$$

and for $\tilde{\tau} \to \infty$ the amplitude B tends to zero like $\exp\left[\frac{1}{2}\Gamma f'(\tau^*)\tilde{\tau}^2\right]$. Thus (14) describes how the quasi-steady solution (7) decays exponentially to zero in a $\{\sigma/\epsilon\}^{\frac{1}{2}}$ neighbourhood of τ^*. Our analysis fails in the case of $f'(\tau^*) = 0$, but the situation is easily remedied by then considering a $\{\sigma/\epsilon\}^{\frac{1}{3}}$ neighbourhood of τ^*. The appropriate amplitude equation is then similar to (11) with $\tilde{\tau}$ replaced by $\tilde{\tau}^2$. The solutions of this equation then enable us to match the quasi-steady solutions either side of $\tau = \tau^*$. However, in this case the quasi-steady solution either side of τ^* is either (7) or (8), and there is now no switch of solutions.

The above discussion can clearly be adapted to any convective- or parallel-flow instability that bifurcates supercritically with increasing Rayleigh or Reynolds number. The results of our calculation are reassuring because they show that the quasi-steady approach used in such problems gives the correct form of the equilibrium solution for most of the time. The approach fails only in a $\{\sigma/\epsilon\}^{\frac{1}{2}}$ neighbourhood of any instant where the flow is locally neutrally stable. However there are cases when the quasi-steady approach will be much less successful. We refer to situations where multiple solutions associated with nearly coincident eigenvalues can occur (see e.g. Davey, DiPrima & Stuart 1968; Hall & Walton 1979). In such situations the equilibrium amplitude solutions predicted by quasi-steady theories are usually determined by solving pairs of coupled cubic amplitude equations. At some values of the appropriate stability parameter more than one stable configuration is possible. In such problems the time interval over which the basic flow variation is important can be comparable to the separation of the eigenvalues, and coupled non-autonomous differential equations with cubic nonlinearities must be solved. The ultimate (if any) stable state set up then depends on the form of the initial perturbation, and can be different from that predicted using a quasi-steady approach.

3. Modulated Couette flow

Suppose that as a particular case we now take $f(\tau) = \cos\tau$ so that we are in a position to discuss the stability of Couette flow modulated at low frequencies. Previous theoretical investigations of this problem have found that modulation has a destabilizing effect on circular Couette flow, whereas experimentally the opposite result has been found.

As a first step towards a reconciliation between theory and experiment it is necessary to describe how Donnelly defined the critical Taylor number for such a flow. Using the ion technique pioneered by himself, Donnelly was able to measure the mean amplitude of any vortex flow superimposed on the basic circumferential flow. The results given by Donnelly typically show an initial linear growth of the amplitude followed by a more rapid growth at higher Taylor numbers. The Taylor number at which this change in the rate of growth of the amplitude took place was defined to be the critical Taylor number. Such a definition is of course suggested by the corresponding steady problem where such a result is now well known. The vortices present when the amplitude grew linearly with increasing Taylor number were described as 'transient vortices' by Donnelly, but we shall see that the low-frequency approach of §2 predicts such vortices.

We return now to the amplitude equation (6) and its equilibrium solution (7) which exists for the part of the period when

$$\cos\tau + \frac{T_1}{2T_0} > 0.$$

This condition simply means that the instantaneous Taylor number of the flow is greater than its steady critical value. It follows that there are three regimes to discuss depending on whether the flow is supercritical for none, part, or all of a period of oscillation of the inner cylinder. The quasi-steady solutions in each of these regimes are as listed below:

(*a*) $\dfrac{T_1}{2T_0} < -1, \quad A^2 = 0.$

(*b*) $-1 < \dfrac{T_1}{2T_0} < 1, \quad A^2 = -\dfrac{\Gamma}{a_1}\left(\cos\tau + \dfrac{T_1}{2T_0}\right), \quad 2n\pi - \tau_1 < \tau < 2n\pi + \tau_1,$

where n is an integer and $\cos\tau_1 = -T_1/2T_0$;

$$A^2 = 0, \quad 2n\pi + \tau_1 < \tau < (2n+1)\pi - \tau_1.$$

$$(c) \quad \frac{T_1}{2T_0} > 1, \quad A^2 = -\frac{\Gamma}{a_1}\left(\cos\tau + \frac{T_1}{2T_0}\right) \quad \text{for all } \tau.$$

In (*b*) above the transition between a finite-amplitude equilibrium state and the zero state takes place in short time intervals of order $\{\sigma/\epsilon\}^{\frac{1}{2}}$ in the manner described in §2. For this reason the solution (*b*) above does not correspond to a periodic solution of the full equations since in an interval where the flow is becoming supercritical the amplitude has a square root singularity at some time. Some further discussion of this point will be given in §4. We further note that the cases $T_1/2T_0 \sim \pm 1$ must be treated as special cases in the manner outlined in §2.

In order to compare our results with those of Donnelly we note that the latter author rescaled the amplitudes that he measured on the amplitude for the unmodulated problem at $\Omega = 5{\cdot}8$. The critical value of Ω for the unmodulated case was found by Donnelly to be $\Omega_c = 5{\cdot}64$, so that in order to determine the implications of (*a*)–(*c*) we redefine these solutions in terms of the amplitude A_D used by Donnelly. We recall that the speed of the inner cylinder is $\Omega(1+\epsilon\cos\tau)$, so that the quasi-steady solutions in the three regimes of interest are as follows.

(I) $\quad \Omega(1+\epsilon) < \Omega_c, \quad A_D = 0.$

(II) $\quad \Omega(1-\epsilon) < \Omega_c < \Omega(1+\epsilon),$

$$A_D = 0{\cdot}74\{\Omega^2(1+\epsilon\cos\tau)^2 - \Omega_c^2\}^{\frac{1}{2}} \quad \text{for} \quad 2n\pi - \tau_1 < \tau < 2n\pi + \tau_1,$$

where $\cos\tau_1 = \epsilon^{-1}(\Omega_c/\Omega - 1)$.

$$A_D = 0, \quad \text{for} \quad 2n\pi + \tau_1 < \tau < 2(n+1)\pi - \tau_1.$$

(III) $\quad \Omega(1-\epsilon) > \Omega_c, \quad A_D = 0{\cdot}74\{\Omega^2(1+\epsilon\cos\tau)^2 - \Omega_c^2\}^{\frac{1}{2}} \quad$ for all τ.

We note that the above expressions are correct only to order ϵ with $\Omega - \Omega_c \sim O(\epsilon)$. The above solutions can be used to compute the mean value of A_D defined by

$$\bar{A}_D = \frac{1}{2\pi}\int_0^{2\pi} A_D\, d\tau, \tag{15}$$

and this is the quantity given in figure 5 of Donnelly (1964). Equation (15) together with (I), (II) and (III) can be used to compute $\bar{A}_D$ for $\Omega - \Omega_c \sim O(\epsilon)$, although the cases $\Omega_c \sim \Omega(1\pm\epsilon)$ should be considered separately. The correction to $\bar{A}_D$ introduced by doing so is negligible, but later we shall at least indicate the result of considering these special cases.

It follows from (15) and (I) that $\bar{A}_D$ is zero unless the maximum instantaneous Taylor number is ever supercritical. When this maximum Taylor number is slightly supercritical in the sense that $\Omega(1+\epsilon) - \Omega_c \ll \epsilon$ we can show that

$$\bar{A}_D \sim \frac{\Omega - \Omega_c}{1+\epsilon},$$

so that initially $\bar{A}_D$ increases linearly with Ω. In contrast with this result we can show that for $\Omega(1+\epsilon) - \Omega_c \gg \epsilon$

$$\bar{A}_D \sim (\Omega - \Omega_c)^{\frac{1}{2}},$$

so that the initial linear growth of $\bar{A}_D$ is ultimately replaced by a square-root growth. We believe that it is this change in the nature of the growth of $\bar{A}$ that is so apparent in the low-frequency results of figure 5 of Donnelly (1964). A direct comparison with Donnelly's results is not feasible because of the reasons described below.

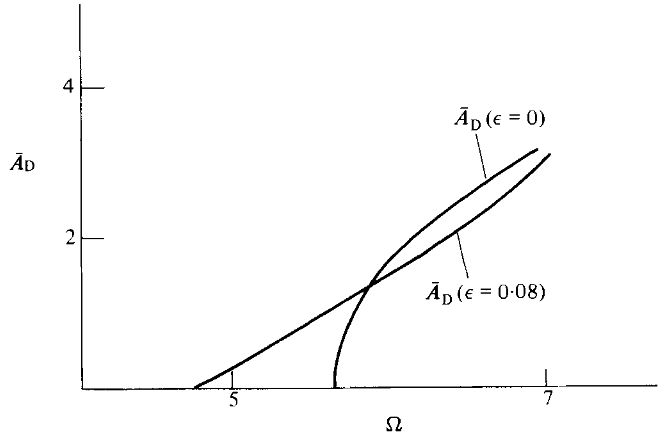

FIGURE 2. The mean amplitude $\bar{A}_D$ for modulated Couette flow with $\epsilon = 0{\cdot}08$.

We refer to the fact that it is well known that in the unmodulated problem the Taylor-vortex flow becomes unstable to wavy vortices at higher Taylor numbers. In the apparatus used by Donnelly the wavy-vortex critical angular velocity is only about 5% greater than the Taylor-vortex critical angular velocity Ω_c. Since the minimum value of ϵ used by Donnelly was $\epsilon = 0{\cdot}08$ it is clear that if at low frequencies the measurements reported by Donnelly correspond to a quasi-steady response of the fluid then wavy vortices cannot be ignored. We do not modify our theory to take this into account but recognize the fact that for such high values of ϵ our results probably only apply to the initial linear growth stage of the vortices. Clearly our results are applicable to the case when wavy vortices never appear during a period of oscillation of the inner cylinder. Such a situation could be realized experimentally by using an apparatus with a wider gap or using a small-gap apparatus at smaller values of ϵ. It should also be pointed out that the characteristic change in the rate of growth of the vortex amplitude with increasing Ω would also be found even in the presence of wavy vortices. Thus, even though our neglect of wavy vortices prevents us reproducing accurately Donnelly's results at low frequencies, we feel that our results and Donnelly's are at least in qualitative agreement. In figure 2 we have used our results to compare $\bar{A}_D$ for different values of Ω with $\epsilon = 0{\cdot}08$. This figure should be compared with the case $P = 46{\cdot}1$ and $\epsilon = 0{\cdot}08$ of Donnelly's figure 5. We note that both figures indicate an initial linear growth in amplitude of the vortex followed by the more usual square-root growth. We feel that such a qualitative agreement is about the most that can be expected in view of our neglect of wavy vortices, which were almost certainly present in Donnelly's experiments.

Finally we return to the special cases $\Omega_c \sim \Omega(1 \pm \epsilon)$, which have so far not been discussed in detail. Consider firstly the case $\Omega_c \sim \Omega(1+\epsilon)$, so that the flow just becomes supercritical for part of the period of oscillation of the cylinder. In this regime it is necessary to consider a time interval of length $O(\sigma/\epsilon)^{\frac{1}{3}}$ centred on the time when the inner cylinder achieves its maximum velocity. The angular velocity Ω is then taken to be such that $\Omega_c - \Omega(1+\epsilon) \sim O(\epsilon^{\frac{1}{3}}\sigma^{\frac{2}{3}})$, and the outcome is that in this time interval the Taylor vortex has amplitude $O(\sigma^{\frac{1}{3}}\epsilon^{\frac{1}{6}})$ and satisfies an equation similar to (11) but with the coefficient of B now quadratic in $\tilde{\tau}$. The solutions of this equation show that any disturbance imposed on the flow in this time interval can only grow at most for part of the interval and ultimately decays to zero. Thus it follows that the results shown in figure 2 are not valid for $\Omega_c - \Omega(1+\epsilon) \sim O(\epsilon^{\frac{1}{3}}\sigma^{\frac{2}{3}})$ and the linear growth of $\bar{A}_D$ discussed above only applies for $\epsilon^{\frac{1}{3}}\sigma^{\frac{2}{3}} \ll \Omega_c - (1+\epsilon) \ll \epsilon$. However, the exact manner in which the linear growth of A_D is altered depends on the precise form of the imposed disturbances. A similar analysis applied to the case $\Omega_c - \Omega(1+\epsilon) \sim O(\epsilon^{\frac{1}{3}}\sigma^{\frac{2}{3}})$ shows that the results of figure 2 are in error by $O(\epsilon^{\frac{1}{3}}\sigma^{\frac{2}{3}})$ in this regime. In view of the smallness of σ assumed in our analysis it follows that such errors are not significant.

4. Comparison with the method of DiPrima & Stuart

We have seen that the finite-amplitude solutions constructed in §§2 and 3 are not periodic in τ if the flow is locally supercritical for only part of each cycle. Moreover in this case the solution in a $(\sigma/\epsilon)^{\frac{1}{2}}$ neighbourhood of the locally neutral time develops a singularity if it is extended for sufficiently large and negative values of $\tilde{\tau}$. Thus any non-zero solution of (11) develops a singularity at some value $\tilde{\tau} = \tilde{\tilde{\tau}}$ where $B \sim (\tilde{\tau} - \tilde{\tilde{\tau}})^{-\frac{1}{2}}$. The physical significance of extending a solution for τ decreasing until this occurs is of course not clear. We presume however that near $\tilde{\tau} = \tilde{\tilde{\tau}}$ a more complicated initial-value problem for bigger disturbances can be formulated which would have $B \sim (\tilde{\tau} - \tilde{\tilde{\tau}})^{-\frac{1}{2}}$ as a long-time behaviour. A similar singularity occurs when the growth of Görtler vortices or Tollmien–Schlichting waves in boundary layers is considered (see Hall 1982*a, b*; Hall & Smith 1982).

We now turn to the question of how the present approach is related to that used by Hall (1975) for the case $\epsilon \ll 1$, $\epsilon/\sigma \sim O(1)$. This limit was investigated using the method of DiPrima & Stuart (1973, 1975), who devised an expansion procedure which they used to investigate the stability to Taylor vortices of flows in journal bearings. In that problem the polar angle ϕ takes the place of the τ-dependence in our problem, and the problems are closely related.

Suppose then that we follow Hall (1975) and consider the limit $\epsilon \to 0$ with $\sigma/\epsilon = \alpha$ held fixed. The Taylor vortex is then of order $\epsilon^{\frac{1}{2}}$ and the appropriate expansion is again (4) but with the $\bar{\tau}$-dependence now dropped. The solution is given in detail by Hall (1975) and we simply note here that the amplitude $A(\tau)$ of the Taylor vortex satisfies

$$\alpha \frac{dA}{d\tau} = \Gamma\left(\cos\tau + \frac{T_1}{2T_0}\right) A + a_1 A^3, \tag{16}$$

where Γ, T_1 and a_1 are as in (6). It is not difficult to show that this equation has periodic solutions for $T_1 > 0$ which can be written in the form

$$A^{-2}(\tau) = -\frac{2a_1}{\alpha} \frac{\int_0^{2\pi} \Psi\, dx + [\Psi(2\pi) - \Psi(0)] \int_0^{\tau} \Psi\, dx}{[\Psi(2\pi) - \Psi(0)]\,\Psi(\tau)}, \tag{17}$$

where

$$\Psi(x) = \exp\frac{\Gamma}{\alpha}\left(2\sin x + \frac{T_1 x}{T_0}\right).$$

We might expect that if the further limit $\alpha \to 0$ with T_1 fixed is taken then our results of §3 would be recovered. This is not the case, but we can recover the results of §3 by solving (16) in the limit $\alpha \to 0$ and relaxing the condition that the solution be periodic in τ.

These solutions can be constructed by following the expansion procedure of §2. We suppose that $-1 < T_1/2T_0 < 1$ and note that the quasi-steady solution of (16) can be written in the form

$$A = \left\{-\frac{\Gamma}{a_1}\left(\cos\tau + \frac{T_1}{2T_0}\right)\right\}^{\frac{1}{2}} + O(\alpha), \tag{18}$$

which is valid in the interval $(-\tau_1, \tau_1)$, where $\tau_1 = \cos^{-1}(-T_1/2T_0)$. In this time interval the flow is locally supercritical and we expect that alternative asymptotic forms for A should be developed near $\pm\tau_1$. In fact near $\tau = \tau_1$ the flow is changing from supercritical to subcritical, and, in terms of the stretched variable $\tilde{\tau} = (\tau - \tau_1)/\alpha^{\frac{1}{2}}$, the appropriate asymptotic solution of (16) that matches with (18) is

$$A = \alpha^{\frac{1}{4}} \left\{ \frac{-\exp\left[-\sin\tau_1 \Gamma \tilde{\tau}^2\right]}{2a_1 \int_{-\infty}^{\tilde{\tau}} \exp\left[-\sin\tau_1 \Gamma \theta^2\right] d\theta} \right\}^{\frac{1}{2}} + O(\alpha^{\frac{1}{2}}). \tag{19}$$

A similar expansion must be set up near $\tau = -\tau_1$, and the appropriate variable is then $\tilde{\tilde{\tau}} = (\tau + \tau_1)/\alpha^{\frac{1}{2}}$. The first term in this expansion satisfies the same differential equation as does the first term in (19), but with τ_1 replaced by $-\tau_1$. The equation must be solved subject to an initial condition at some $\tilde{\tilde{\tau}} = \tilde{\tilde{\tau}}^*$ and is required to match with (18) when $\tau \to -\tau_{1+}$. The required solution develops a singularity at some $\tilde{\tilde{\tau}} < \tilde{\tilde{\tau}}^*$ and so there is no periodic solution of (16) when $-1 < T_1/2T_0 < 1$; nevertheless we believe that in the limit $\alpha \to 0$ such disturbances are the most likely to be observed experimentally. We can of course choose to restrict our attention to periodic solutions of (16), and it is interesting to see how such solutions are related to the class of disturbance discussed above.

The periodic solutions of (16) were given earlier by (17), and their asymptotic form when $\alpha \to 0$ is readily obtained by standard techniques. Alternatively the appropriate asymptotic forms can be found by constructing asymptotic solutions of (16) directly. For the sake of brevity we shall here summarize briefly the outcome of either calculation.

Firstly if $T_1/2T_0 > 1$ the quasi-steady asymptotic form (18) is valid for all τ, and so is the required periodic solution of (16). If $T_1/2T_0$ is decreased below unity we expect that (18) will describe the periodic solution for only a limited range of values of τ. Thus if we take $-1 < T_1/2T_0 < 1$ the expansion (18) is found to be valid in the interval (τ_2, τ_1), where $\tau_2 \in (-\tau_1, \tau_1)$ is to be determined.

Near τ_1 we retain (19), which describes the initial decay of the Taylor vortex when the flow becomes subcritical. If we allow $\tilde{\tau}$ to tend to ∞ in (19) we find

$$A \sim \alpha^{\frac{1}{4}} \exp\left[\frac{-\sin\tau_1 \Gamma(\tau-\tau_1)^2}{2\alpha}\right] (-2a_1)^{-\frac{1}{2}} (\pi\Gamma \sin\tau_1)^{\frac{1}{4}}, \tag{20}$$

so that for $\tau > \tau_1$ and $\tau - \tau_1 \sim O(1)$ we expect $A \sim \alpha^{\frac{1}{4}} e^{-\theta(\tau)/\alpha}$ for some θ. Thus when τ increases through τ_1 the amplitude of the vortex first falls off from $O(1)$ to $O(\alpha^{\frac{1}{4}})$ and ultimately to $O(\alpha^{\frac{1}{4}} e^{-\theta(\tau)/\alpha})$. Since we are interested in periodic solutions of (16) we follow the development of this exponentially small disturbance until it ultimately grows sufficiently for nonlinear effects to be important. This development must of course be given by solving the linearized form of (16) to give

$$A = \alpha^{\frac{1}{4}} \exp\left[\frac{\Gamma}{\alpha}\left(\sin\tau + \frac{T_1\tau}{2T_0}\right) - \frac{\Gamma}{\alpha}\left(\sin\tau_1 + \frac{T_1\tau_1}{2T_0}\right)\right] (-2a_1)^{-\frac{1}{2}} (\pi\Gamma \sin\tau_1)^{\frac{1}{4}} \tag{21}$$

This solution matches with (20) when $\tau \to \tau_{1+}$ and is valid in the range $\tau_1 < \tau < 2\pi + \tau_2$, where τ_2 is still to be determined.

The solution (21) decays until $\tau = 2\pi - \tau_1$, beyond which it will grow until $A \sim O(\alpha^0)$, when the exponential in (24) is $O(\alpha^{-\frac{1}{4}})$. This condition determines τ_2, which therefore satisfies

$$\frac{\Gamma}{\alpha}\left(\sin\tau_2 + \frac{T_1\tau_2}{2T_0}\right) - \frac{\Gamma}{\alpha}\left(\sin\tau_1 + \frac{T_1\tau_1}{2T_0}\right) = -\tfrac{1}{4}\ln\alpha.$$

We can show from above that τ_2 is given by

$$\tau_1 = \tau_{20} - \frac{\alpha \ln \alpha}{4\left(\cos \tau_{20} + \dfrac{T_1}{2T_0}\right)} + \ldots,$$

where
$$\int_{\tau_1}^{\tau_{20}+2\pi} \left(\cos \tau + \frac{T_1}{2T_0}\right) d\tau = 0. \tag{22}$$

The latter condition determines τ_{20} in $(-\tau_1, \tau_1)$ only if $T_1 > 0$, so that periodic solutions of (16) exist only for $T_1 > 0$.

In order to complete the construction of the periodic solution of (16) we consider a time interval of length $O(\alpha)$ near $\tau_{2+2\pi}$. We thus write

$$\bar{t} = \frac{\tau - \tau_2}{\alpha}$$

and expand A in the form
$$A = \bar{A}_0(\bar{t}) + O(\alpha \ln \alpha). \tag{23}$$

The function $\bar{A}_0$ is then found to satisfy the equation

$$\frac{d\bar{A}_0}{d\bar{t}} = \Gamma\left(\cos \tau_{20} + \frac{T_1}{2T_0}\right) \bar{A}_0 + a_1 \bar{A}_0^3,$$

and a solution of this equation is

$$\bar{A}_0^2 = \frac{\exp\left[-2\Gamma\left(\cos \tau_{20} + \dfrac{T_1}{2T_0}\right) \bar{t}\right]}{(-2a_1)^{-\frac{1}{2}} (\pi\Gamma \sin \tau_1)^{-\frac{1}{2}} + \dfrac{a_1 \exp\left[2\Gamma(\cos \tau_{20} + T_1/2T_0)\bar{t}\right]}{\Gamma(\cos \tau_{20} + T_1/2T_0)}}. \tag{24}$$

This function enables us to match (23) when $\bar{t} \to -\infty$ with (21) when $\tau \to \tau_{2-} + 2\pi$. In addition we see that when $\bar{t} \to \infty$ the function $\bar{A}_0$ equilibrates to the value of the quasi-steady solution (19) evaluated at τ_{20}. Thus we have constructed a periodic solution of (16). We note that the two solutions coincide for only part of the time when the flow is locally supercritical. If $T_1/2T_0$ increases beyond unity the solutions coincide for all time and are given by the quasi-steady form (19). If $T_1/2T_0$ is in the range $(-1, 0)$ then only the non-periodic solution exists.

The above discussion shows clearly the structure of the non-periodic and periodic solutions of (16) in the limit $\alpha \to 0$. Clearly the analysis of §3 is easily modified to construct directly the periodic Taylor-vortex flows appropriate to the limit $\epsilon \to 0$, $\sigma \ll \epsilon$. Again it is found that the periodicity conditions applied in the regime where the flow is supercritical for only part of a cycle causes the amplitude of the disturbance to remain small when the flow becomes supercritical. In view of the obvious similarity of this case with that discussed above we do not discuss it further. We note that, even though this periodic solution is probably of no physical relevance in the modulated-Couette-flow problem, the necessity of retaining the periodicity condition in for example the journal-bearing problem means that such periodic solutions are important elsewhere. Thus the asymptotic form of the periodic solution given above is applicable to the journal-bearing problem and shows how the structure found by DiPrima & Stuart is modified in the WKB limit.

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